Subsequences.

21/03/2014

1, 2, 3, 4, 5, 6, 7, 8, 9, ...

Def. Suppose $(a_n)_{n=1}^{\infty}$ is a seq. and $n_1, n_2, n_3, \dots$ is a seq. of natural numbers s.t. $n_1 < n_2 < n_3 < n_4 < \dots$. The sequence $(a_{n_k})_{k=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$.

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad \dots$

$n_1 = 2 \quad n_2 = 4 \quad n_3 = 5, \quad n_4 = 7 \quad \dots$

$a_{n_1} \quad a_{n_2} \quad a_{n_3} \quad \dots$

$a_2 \quad a_4 \quad a_5 \quad \dots$

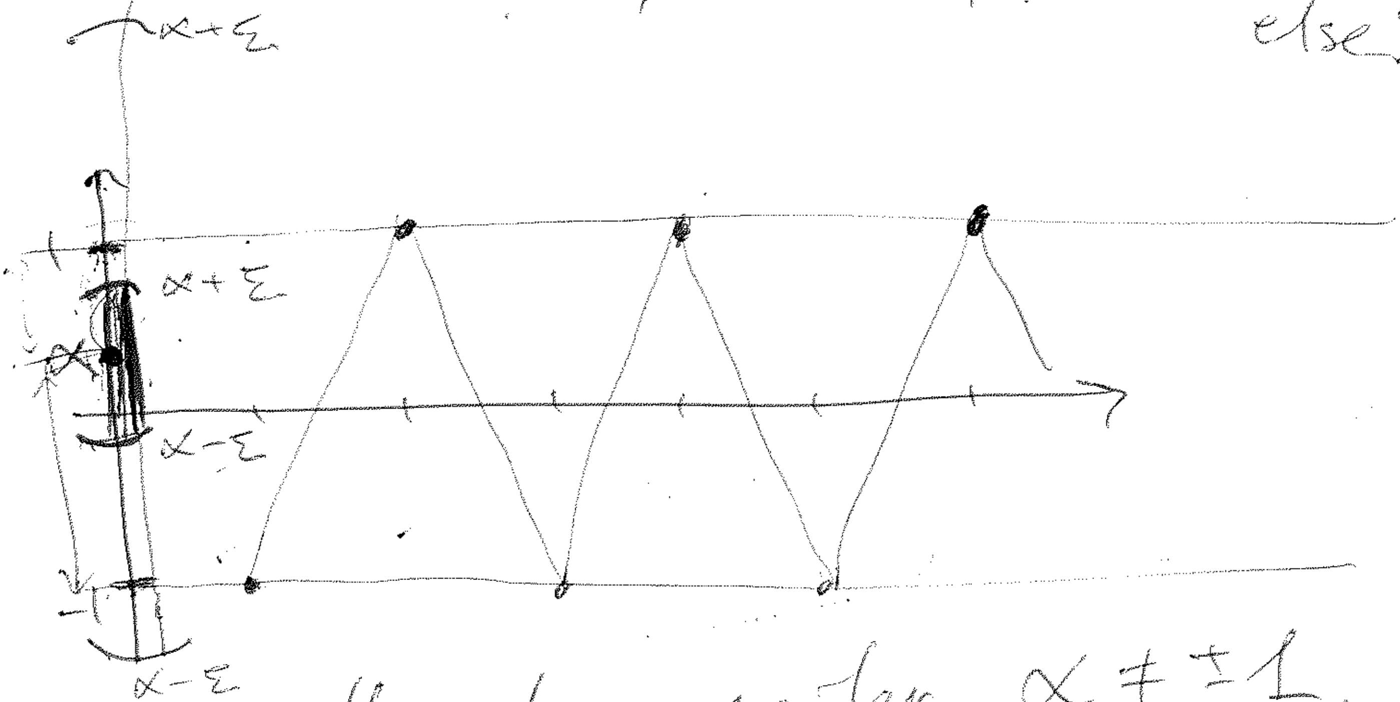
Ex. $a_n = (-1)^n$, $-1, 1, -1, 1, -1, 1, -1, 1, -1, 1$

Then $(a_{2n})_{n=1}^{\infty}$ is a subseq. $1, 1, 1, \dots$

$(a_{3n})_{n=1}^{\infty}$ is a subseq. $-1, 1, -1, \dots$

Let us prove that $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^n$ diverges.

Assume $\lim_{n\to\infty} (-1)^n = \alpha$. There are three cases: $\alpha = 1$, $\alpha = -1$, α is something else



We will only consider $\alpha \neq \pm 1$.
The other two cases analogously.
By def., ~~for~~ for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ s.t. $n > N \Rightarrow |a_n - \alpha| < \varepsilon$.

Choose $\varepsilon = \min\left\{ \frac{|\alpha - 1|}{2}, \frac{|\alpha - (-1)|}{2} \right\}$.

~~Then for every n, we have~~

$\cancel{|a_n - \alpha| < \varepsilon \Rightarrow |a_n - \alpha| < \frac{|\alpha - 1|}{2}}$

$\cancel{\Rightarrow |a_n| - |\alpha| < \frac{|\alpha - 1|}{2}}$

$\cancel{\Rightarrow |a_n| < |\alpha| + \frac{|\alpha - 1|}{2}}$

$\cancel{|a_n - \alpha| < \frac{|\alpha - 1|}{2}}$

$\varepsilon + \alpha < a_n < \varepsilon + \alpha$

$-\frac{|\alpha-1|}{2} + \alpha < a_n < \frac{|\alpha-1|}{2} + \alpha$

$-\frac{\alpha-1}{2} + \alpha < a_n$ if $\alpha \geq 1$

$\frac{\alpha}{2} + \frac{1}{2} <$

If $a_n = 1$, then

$|\alpha - a_n| = |\alpha - 1|$ ~~and~~ $> \frac{|\alpha-1|}{2}$

$\geq \varepsilon$.

If $a_n = -1$, then

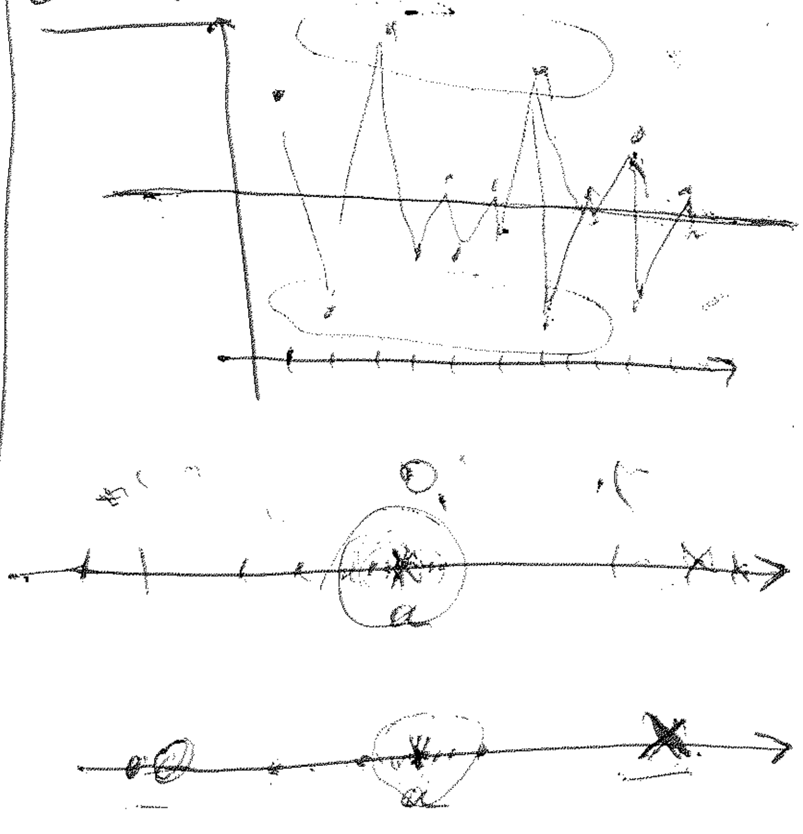
$|\alpha - a_n| = |\alpha - (-1)| > \frac{|\alpha-(-1)|}{2} \geq \varepsilon$.

Thus, $|\alpha - a_n| < \varepsilon$ fails for all n. $\square$

Contradiction.

Lec 10

Def. Suppose $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$. The point $a \in \mathbb{R}$ is a cluster point of $(a_n)_{n=1}^{\infty}$ if for every $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon$.



Is this same as limit? NO.

Example. $\{0, 1, -1, 0, 1, -1, 0, 1, -1, \dots\}$

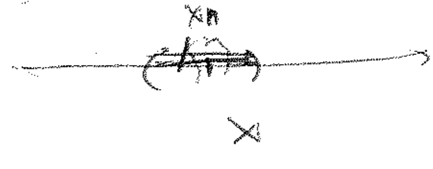
Cluster points: $0, 1, -1$.

Note. $x = \limsup_{n\to\infty} a_n \Leftrightarrow x$ is the greatest cluster point of $(a_n)_{n=1}^{\infty}$. Similarly, liminf.

Proof. Exercise.

Theorem. Assume $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ and $x \in \mathbb{R}$.

(i) x is a cluster point of $(x_n)_{n=1}^{\infty}$ iff for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $n \geq N$ s.t. $|x_n - x| < \varepsilon$.

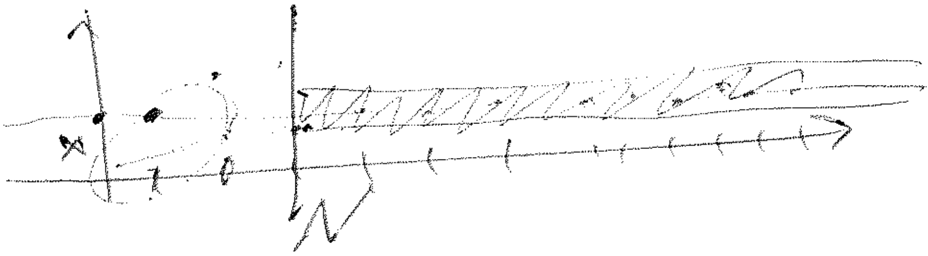


(ii) x is a cluster pt. $\Longleftrightarrow$ there exists $(x_{n_k})_{k=1}^{\infty}$ s.t. $\lim_{k\to\infty} x_{n_k} = x$.

(iii) $\lim_{n\to\infty} x_n = x$ iff every subsequence of $(x_n)_{n=1}^{\infty}$ converges to x.

(iv) $(x_n)_{n=1}^{\infty}$ converges iff $(x_n)_{n=1}^{\infty}$ is bounded and has exactly <u>one</u> cluster point.

<u>Proof.</u> (i) $(\Rightarrow)$ Assume x is a cluster pt. Then, given $\varepsilon > 0$, there exist infinitely many x_n s.t. $|x - x_n| < \varepsilon$. At least one of these x_n must be such that $n \geq N$.

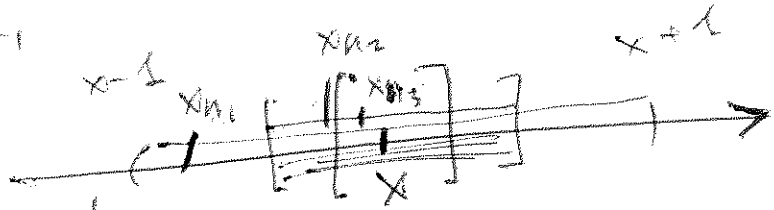


$(\Leftarrow)$ Fix $\varepsilon > 0$. For $N = 1$, there is $n_1 \geq 1$ s.t. $|x_{n_1} - x| < \varepsilon$. Now, for $N = n_1 + 1$, there is $n_2 \geq n_1 + 1$ s.t. $|x_{n_2} - x| < \varepsilon$.

Now, for $N = n_2 + 1$, there is n_3 s.t. ...
Continue this process. Get a seq. $(x_{n_k})_{k=1}^{\infty}$ s.t. $|x_{n_k} - x| < \varepsilon$ for all k.
Thus, x is a cluster pt.

(ii) ($\Rightarrow$) x is cluster. Then for $\varepsilon = 1$, ~~there exist N_1 s.t. $n \geq N_1$ $\Rightarrow |x_n - x| < \varepsilon$~~ there exist infinitely many x_n's s.t. $|x_n - x| < 1$. Take one of them, call it x_{n_1}. Then, by (i), for $\varepsilon = \frac{1}{2}$ and $N = n_1$, there is $n_2 > n_1$ s.t. $|x - x_{n_2}| < \frac{1}{2}$.

For $\varepsilon = \frac{1}{3}$ and $N = n_2$, there is n_3 s.t. $n_3 > n_2$ and $|x_{n_3} - x| < \frac{1}{3}$.



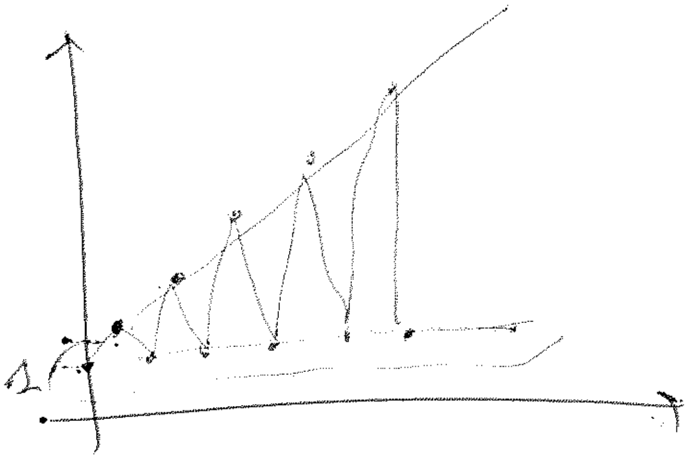
Continuing this, we obtain a subseq. $(x_{n_k})_{k=1}^{\infty}$. It is easy to show this subseq. converges to x.

($\Leftarrow$) If there is a subseq. $(x_{n_k})_{k=1}^{\infty}$ s.t.

$\lim_{k\to\infty} x_{n_k} = x$, fix $\varepsilon > 0$.

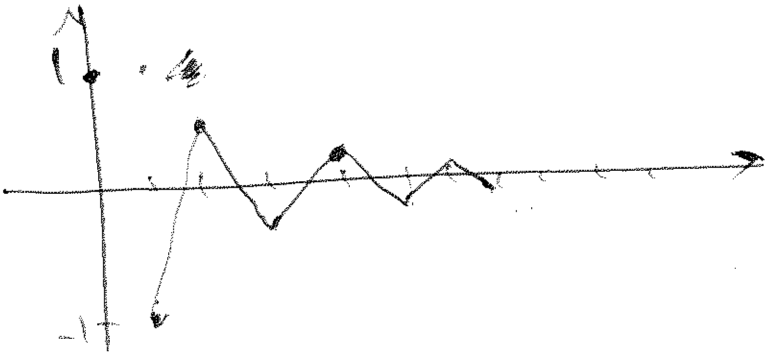
There is $N \in \mathbb{N}$ s.t. $k \geq N \Rightarrow |x_{n_k} - x| < \varepsilon$. But there are infinitely many such k. Therefore, there are infinitely many x_{n_k} in the ε-nbhd of x. $\square$

(iii), (iv) Exercises. $\square$



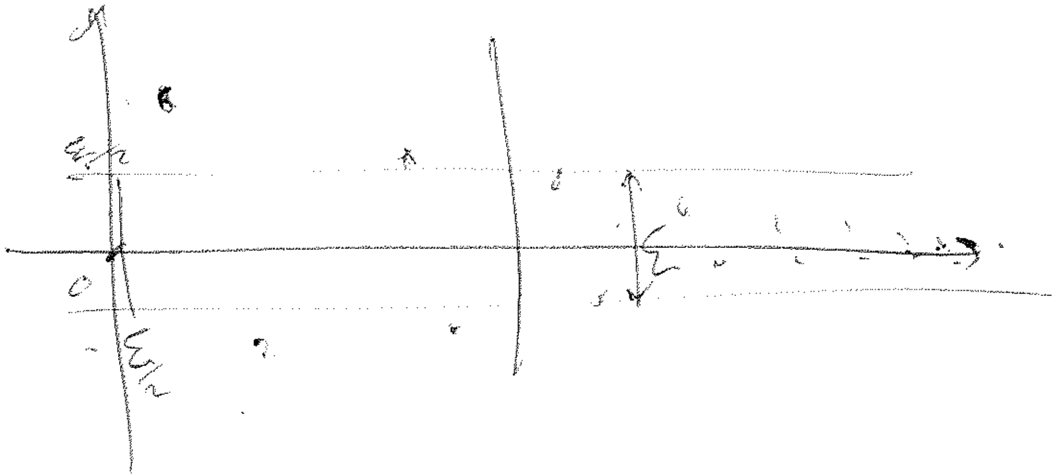
<u>Example.</u>

$x_n = (-1)^n \frac{1}{n}$.



<u>Cauchy sequences.</u>

<u>Def.</u> $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ is a Cauchy seq. if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $n, k \geq N \Rightarrow |x_n - x_k| < \varepsilon$.



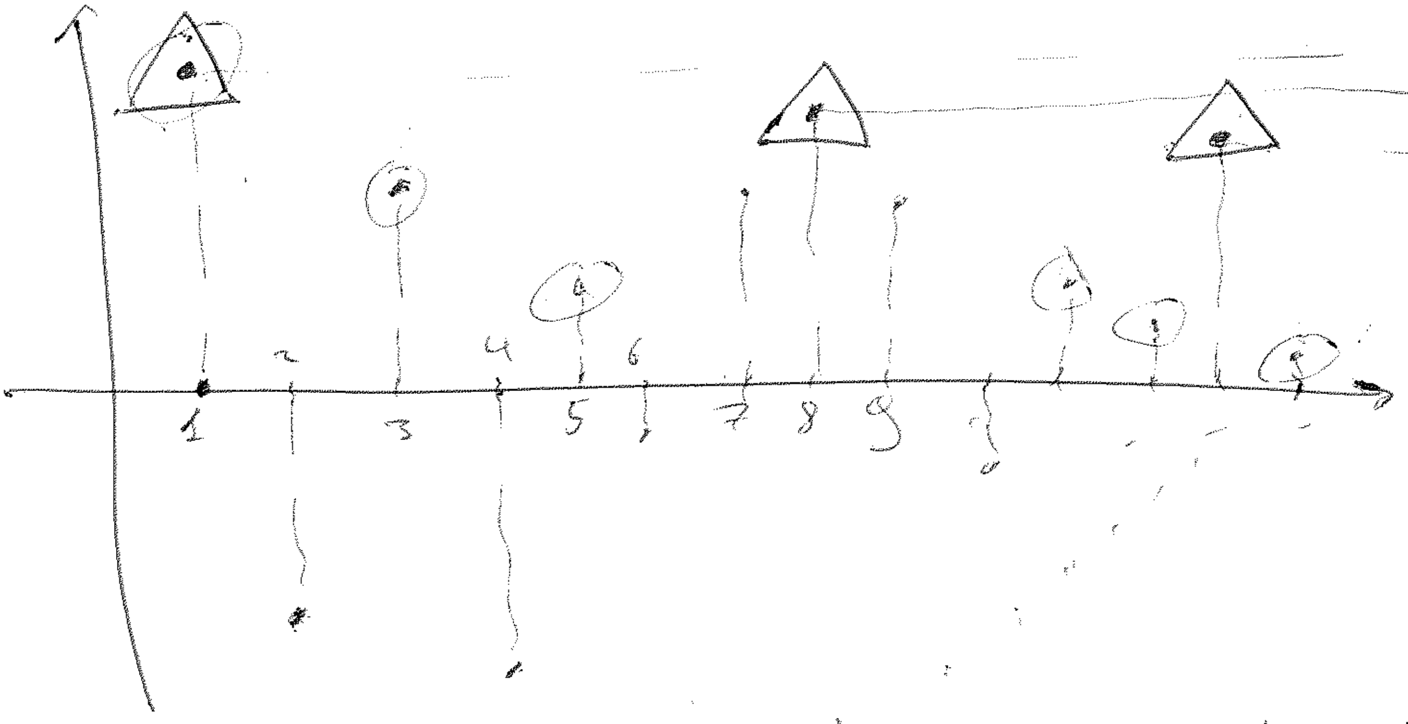
<u>Theorem.</u> If $(x_n)_{n=1}^{\infty}$ converges, then x_n is Cauchy.

<u>Proof.</u> Let $x = \lim_{n\to\infty} x_n$. Fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$. Then, if $n, k \geq N$, then

$$|x_n - x_k| = |x_n - x + x - x_k|$$
$$\leq |x_n - x| + |x - x_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

<u>Lemma.</u> Every sequence in $\mathbb{R}$ has a monotone subseq. Lec 11



<u>Proof.</u> We say $k \in \mathbb{N}$ is a <u>peak point</u> of $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ if for all $n > k$, $a_n < a_k$.

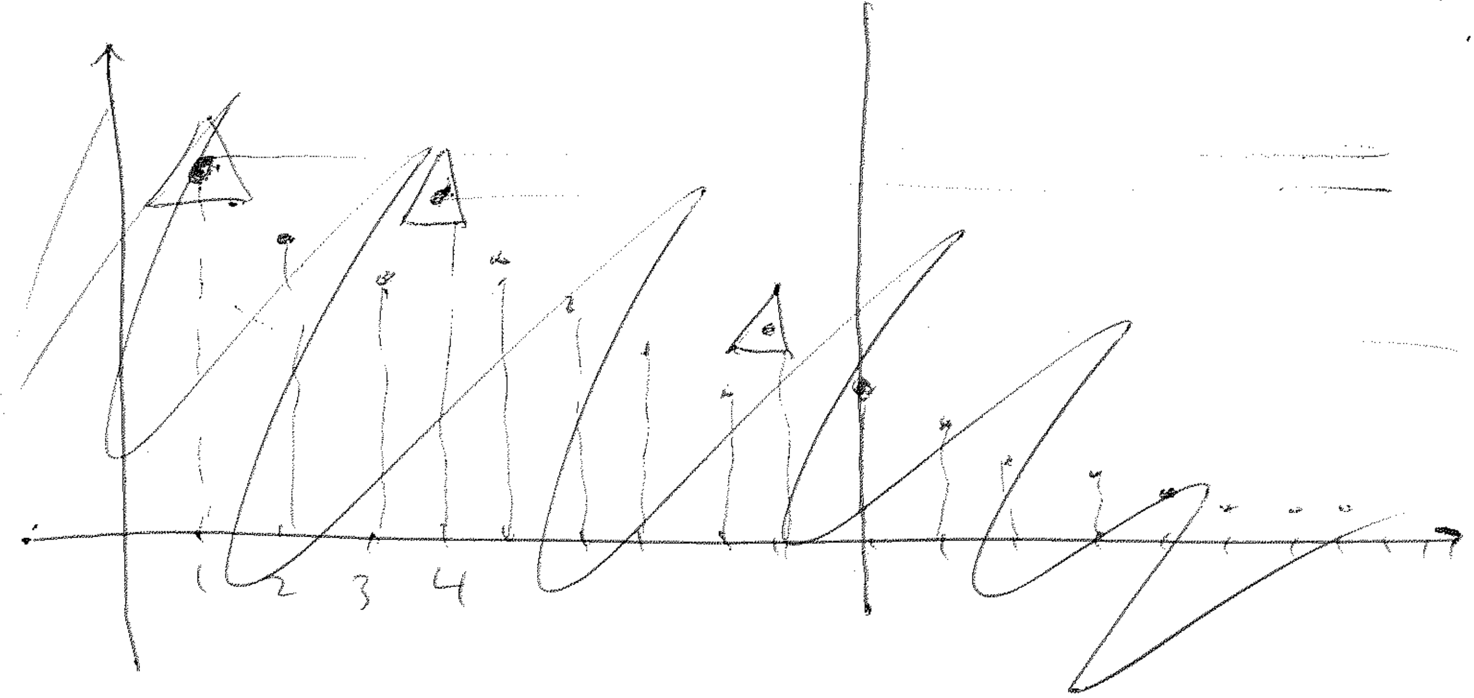
<u>Case 1.</u> There are infinitely many peak points, $n_1 < n_2 < n_3 < n_4 < \cdots$.

Then, by def.,

$$a_{n_1} > a_{n_2} > a_{n_3} > a_{n_4} > \cdots$$

Thus, $(a_{n_k})_{k=1}^{\infty}$ is monotone decr. we're done.

<u>Case 2.</u> There are finite many pear points $k_1, k_2, \dots, k_m$.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